MATH4050 Real Analysis
Assignment 7

There are 5 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

1. (3rd: P.89, Q3)

   Let f be a nonnegative measurable function. Show (by the monotonicity) that $\int f = 0$ implies $f = 0$ a.e. , and that $\int f \in \mathbb{R}$ implies that $f(x) \in \mathbb{R}$ a.e.

2. (3rd: P.89, Q4; 4th: P.85, Q24)

   Let f be a nonnegative measurable function.

   a. Show that there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions each of which vanishes outside a set of finite measure such that $f = \lim \varphi_n$.

   Let $E \in \mathcal{M}$, $\mathcal{S}_1 = \{\varphi : \varphi \in \mathcal{S}_0^+(E), \varphi \leq f\}$

   b. Show that $\int f = \sup \int \varphi$ over all nonnegative simple functions $\varphi \leq f$ with φ vanishing outside a set of finite measure. $\mathcal{S}_2 = \{\varphi : \varphi \in \mathcal{S}^+(E), \varphi \leq f\}$.

   Show that $\int_E f = \sup\{\varphi : \varphi \in \mathcal{S}_1\} = \sup\{\varphi : \varphi \in \mathcal{S}_2\}$

3. (3rd: P.89, Q5)

   Let f be a nonnegative integrable function. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f$$

   is continuous by using Theorem 10 (3rd. ed).
   (Note: Theorem 10 is the monotone convergence theorem)

4. * (3rd: P.89, Q6; 4th: P.85, Q25)

   Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converge to f, and suppose $f_n \leq f$ for each n. Show that

$$\int f = \lim \int f_n.$$

5. * (3rd: P.89, Q7; 4th: P.85, Q25 for part b.)

   a. Show that we may have strict inequality in Fatou's lemma. (Consider the sequence $\{f_n\}$ defined by $f_n(x) = 1$ if $n \leq x < n+1$, with $f_n(x) = 0$ otherwise). • Is the lemma true if relax the condition $0 \leq f_n$ to $-1 \leq f_n$?

   b. Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions. (Let $f_n(x) = 0$ if $x < n$, $f_n(x) = 1$ for $x \geq n$) • How about on $[a,b]$ rather than on $\mathbb{R}$?

6.* Show that $[0\,1) = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$. Let $N \in \mathbb{N}$ and

$$f_N(x) = \begin{cases} n \cdot 2^n & \text{if } x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \text{ for some } n \geq N \\ 0 & \text{otherwise.} \end{cases}$$

Any relevant of $\{f_N : N \in \mathbb{N}\}$ in relation to Q5.